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Editorial

LEONHARD EULER AND HIS CONTRIBUTION TO THE DEVELOPMENT OF MECHANICS (On the 300th anniversay of his birth)☆

In April 2007 we celebrate the 300th anniversary of the birth of Leonhard Euler, one of the greatest mathematicians and mechanics of all time. It is largely through Euler's work that mathematical analysis (including the theory of differential equations and the calculus of variations) came into being, that general mechanics, rigid body mechanics and the hydrodynamics of an ideal fluid became established and that the language and style of modern scientific literature was created.

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1. A brief biography of Euler

Leonhard Euler was born on 15 April 1707 (here and below, all dates are according to the Gregorian calendar) in Basel (Switzerland). It was there that he received his initial education and graduated from university. In Basel, Euler's ability attracted the attention of the elderly Johann Bernoulli, who undertook to supervise his independent studies and suggested that he visit him weekly at home to tackle the most weighty problems. In 1726, Bernoulli wrote of "The happiest talent of the young Leonhard Euler, from whose penetrative and sharp mind we are expecting the greatest things". Euler then became aquainted with the Bernoulli family, the two eldest sons of which were invited in 1725 to join the Petersburg Academy of Sciences founded by Peter the Great. Shortly after, under the patronage of the Bernoulli brothers, Euler was also invited to Petersburg, and arrived there on 24 May 1727 and immediately became actively involved in academic life. He presented his first paper at the beginning of August 1727, and from that time on the stream of studies by him continued until the end of his life. In the 1730s he spoke at meetings more often than any other – on average 10 times a year (a total of 30–40 papers a year), twice as often as those academicians who were to follow him. In the Academy of Sciences journal **Commentarii** alone, Euler published 58 papers in the period from 1730 to 1740. In January 1731 he became Professor of Theoretical and Experimental Physics and after a further three years was given the chair of the Faculty of Higher Mathematics, which he occupied for 8 years until he left for Berlin. From 1735 onwards he carried out a great deal of work in the Geographical Department of the Academy on the preparation of a general map of Russia, and for many years he carried out systematic observations of the sun in the astronomical observatory. As his authority grew, so too did his material well-being.

The Petersburg Academy played a key role in establishing Euler as a scientist of world renown. He remained content with the working conditions in St Petersburg right up until 1740, when the Empress Anna Ioannovna died and the prosecution of many statesmen, including the president of the Academy, began.

Meanwhile, Friedrich II, who had come to the Prussian throne, conceived the idea of creating in Berlin a brilliant academy of sciences, and as early as the summer of 1740 he saw to it that the "great algebraist" Leonhard Euler was offered a post. Taking into account the unstable situation in Petersburg, Euler accepted the offer. Permission to leave Petersburg was granted to him on preferential terms, with the fixing of a pension and appointment as an honarary member of the Petersburg Academy. He left Petersburg, and on the 25 July 1741 arrived in Berlin. An attempt by him to take part in the organization of the Academy did not meet with the support of the King, who showed preference to French enlighteners. On 3 February 1746, Euler was appointed director of the mathematics "class" of the Berlin Academy and he remained in this position for 20 years. As in Petersburg, he presented about 10 scientific papers a year at weekly meetings of the Academy and was recruited to sit on various technical expert commissions. In spite of the fact that he played an important role in the Berlin Academy, in 1764–1765 disagreements arose with his fellow academicians and with Friedrich II on financial matters, and on 28 July 1766 Euler returned to Petersburg, where he had been constantly invited.

It must be stressed that, for the 25 years of his most active creative life in Berlin, Euler maintained constant ties with the Petersburg Academy of Sciences. In these 25 years he published about 130 scientific papers in Berlin and 100 papers in Petersburg, not counting individual books. In Berlin, Russian scientists seconded to Euler from Petersburg came to stay with him and gain practical experience. As before, he was recruited to sit on various kinds of expert commission and to solve complex scientific and internal academic problems in Russia. The support he gave to the first studies of Lomonosov is well known, and it was largely with Euler that the responsibility lay to select specialists for the Petersburg Academy of Sciences and to represent its interests in Central Europe. In 1760, Euler wrote to Russia: "I have worked for the Imperial Academy hitherto not as an absent member but as if I had been there in person".

In 1766, Euler was received in Petersburg with great honour. However, as a result of an unexpectedly severe loss of sight, by the autumn of 1766 he was no longer able to read and write and was forced to rely for this on help from others. Nonetheless, in the 17 years remaining, he carried out an improbable amount of research, and in these years he prepared almost half the total number of his published papers (although they did not contain, generally, such thorough-going results as characterized his previous work). In the 1770s, as before, he continued to be recruited to sit on various expert commissions; thus, in 1776 he was a member of a commission set up to examine plans for a bridge over the river Neva, drawn up by I. P. Kulibin.

On the 18 September 1783, the 76-year-old Euler, as always, was engaged in mathematical research, talked after dinner about the discovery of a seventh planet, in the evening after tea joked with his grandchild and then, unexpectedly, with the words "I am dying", lost consciousness and, after a few hours, died. The Petersburg Academy of Sciences paid him due honour, and in 1786 a bust of him on a marble column was placed in the conference hall of the Academy, opposite the president's chair. However, subsequently, Euler's grave was lost. It was only after 50 years that it was again found by chance, and in 1837 a grand granite stone with the modest inscription **Leonhardo Eulero Academia Petropolitana** was laid upon it. On the occasion of the 250th anniversary of his birth, his remains and gravestone were removed to the necropolis at the former Lazarevskii cemetery at the Alexander Nevskii monastery.

2. The principles of mechanics and mathematical analysis

The contributions by Euler to the development of rational mechanics and mathematical analysis (including the theory of differential equations and the calculus of variations) are most significant. Euler's papers on mechanics comprise roughly one-third of all his work. It must be emphasized that mechanics was Euler's first serious passion. In his intact "notebooks", which he wrote at the age of 18–20, there are already detailed plans of conceived general treatises on point dynamics and the theory of the flow of fluids (and also on music theory). The lively interest he showed in mechanics is also indicated by a comparative analysis of his published essays written in the first 10 years of his scientific creativity (1726–1735). Of their total volume, numbering about 1800 pages (the pages of *Opera omnia*), almost two-thirds is devoted to mechanics and only one-quarter to higher mathematics.

The principal initial concepts of mechanics and the laws of motion were summed up and clearly formulated in Newton's 1687 publication **Philosophiae naturalis principia mathematica**. However, many important elements were missing in Newton's work, above all those for constructing system, rigid body and continuum mechanics. The main obstacle remaining was that the **Principia** were set out using the geometrical method of the ancients, which did not open up ways for subsequent analysis. The development of a systematic approach to solving problems and also to extending of the principles of dynamics themselves were held back by the absence of a well-developed general analytical exposition. The first attempt to set out the entire dynamics in a slightly simpler and more systematic form, using the elements of mathematical analysis, was undertaken by Jacob Hermann in his **Phoronomia** (1715/1716).

Initially, the young Euler directed his efforts to ordering the entire point dynamics at that time and its successive transposition into the language of mathematical analysis. He did this in the two-volume **Mechanica** (E.15, 16; 1736) [here and below, Euler's works are given numbers (with the prefix E) according to the well-known index compiled by Gustaf Eneström]. Mathematical methods were then, for Euler, inextricably linked with mechanics problems for whose solution they were used. Before Euler, the foundations of differential and integral calculus had already been laid. Mathematics and mechanics were confronted with the task of the comprehensive development of this calculus and its use in various applications. The rich potential of mathematical analysis needed to be exploited in order to raise the theoretical and applied divisions of higher mathematics and mechanics from the state of a combination of individual expert methods and solved problems to the level of a systematically constructed science. Euler's creative genius was mainly devoted to solving this problem. He prepared a six-volume classical course of mathematical analysis, about 3000 pages long, two volumes of which were devoted to **Introductio in analysin infinitorum** (E.101, 102; 1748), one volume to **Institutiones calculi differentialis eum ejus usu in analisi finitorum ac doctrina serierum** (E.212; 1755) and three volumes on **Institutiones calculi integralis** (E.342–366, 385; 1768–1770). There was nothing like to this brilliant course, translated in our time into Russian, among essays of the eighteenth century. Many of the results set out there were obtained by Euler himself and entered the golden treasury of mathematical analysis. Mention must be made of his treatise on the calculus of variations, entitled **Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes** (E.65; 1744), and also the two-volume **Einleitung zur Rechew-Kunst** (E.17, 35; 1738–1740) for the Petersburg Academic Gymnasium, and the two-volume **Vollständige Anleitung zur Algebra** (E.387, 388; 1768–1769), which includes the theory of algebraic equations and Diophantine analysis, and which has been printed in about 30 editions in six European languages.

Among the specific results in mathematical analysis he obtained, a special role in the development of mechanics was played by methods he developed for solving linear differential equations with constant coefficients, and also certain solutions of linear differential equations with variable coefficients, particular cases of which are the Bessel and Legendre equations and the hypergeometric equation. Euler proposed methods for the approximate integration of differential equations. He established general approaches in the calculus of variations and specified a differential equation (subsequently called Euler's equation) defining the conditions of a functional extremum. He began to investigate a number of important special functions (the B- and Γ- functions, Bessel functions of the first kind, and the ζ-function of a real argument) and made a fundamental contribution to the development of the theory of analytical functions and number theory. He is also responsible for the wide introduction of many now generally accepted mathematical notations (such as π, e, i, inverse trigonometric functions, etc.).

Euler's own, new, distinct style of exposition helped his work to influence the subsequent development of science. The main results he obtained can be used today in their original form. He is probably the only eighteenth-century scientist whose work can be easily read today. In his old age, Lagrange said: "True enthusiasts should always read Euler, because in his papers everything is clear, well-stated and well-calculated, because they are rich in excellent examples, and it is always necessary to learn from primary sources". In his **Traité de mécanique céleste**, Laplace wrote that Euler, "owing to his discoveries in all areas of analysis and the refinement he introduced into the language, can be considered the father of modern analysis". Ostrogradskii, in this regard, observed: "This title of father is quite merited, since it was Euler who created modern analysis and invented today's mathematical language".

After this general digression, we will return to Euler's **Mechanica**. In a footnote to the first chapter of the first volume of **Mechanica**, he gave an overview of mechanics as it appeared to him in the middle of the 1730s: "Firstly we will examine infinitesimal bodies, i.e. those that can be regarded as points. We will then deal with bodies having a finite size – those that are rigid, not allowing their shape to be changed. Thirdly, we will discuss flexible bodies. Fourthly, we will discuss bodies that allow of tension and compression. Fifthly, we will investigate the motion of many separate bodies, one of which is preventing the others from executing motions in the way they are inclined to do. Sixthly, the flow of fluid bodies will be examined. In relation to these bodies, we will not only examine how, left to themselves, they continue to flow, but also investigate how external factors, i.e. forces, act on these bodies".

In carrying out this huge programme, Euler played a most significant role in developing the principles of rigid body dynamics and the dynamics of an ideal fluid, to which he gave a form similar to that familiar to us from today's textbooks (we need only recall Euler's classical equations in rigid body kinematics and dynamics and in hydrodynamics). He also made a considerable contribution to the development of the oscill theory and the mechanics of flexible and elastic bodies. The development of a general theory of elasticity was to take place in the nineteenth century.

Before talking about his individual results in mechanics, we will make one general observation concerning the place Euler's work occupies among that of his contemporaries. Of the first-class mechanics of a similar age to Euler, mention must be made of Daniel Bernoulli, Clairaut and d'Alembert. d'Alembert was undoubtedly the greatest of these, and Lagrange stood out among Euler's younger contemporaries. However, Lagrange and his **Mécanique analytique** personify a stage in the mathematical development of mechanics that followed Euler. Complex relations built up between d'Alembert and Euler. d'Alembert's researches overlapped by Euler's work in practically all areas of mechanics. They interweaved particularly closely in rigid body dynamics, in approaches to the construction of the dynamics of an ideal fluid, and in string vibration theory. d'Alembert was, undoubtedly, a brilliant rival to Euler in mechanics. His outstanding ideas were often ahead of Euler's investigations, but d'Alembert used an outdated, ponderous mathematical language, and his ideas were expressed in an inaccessible, unclear form. Euler, on the other hand, gave a new rigorous style to the exposition of the exact sciences. As a result, for most of the work where Euler and d'Alembert were in competition, posterity has kept with Euler.

3. The mechanics of systems and celestial mechanics

In his **Mechanica**, Euler was the first to set out systematically the dynamics of a free point mass and a point lying on a prescribed curve or surface. He carried out a successive study of the motion of a point in the absence of resistance (in a void) and in a resistant medium. The investigation was conducted in natural coordinates connected with the trajectory of motion. Speaking of the mechanics of a system, it must be pointed out that Euler did not devote a single independent study to this as such, although, of course, he repeatedly examined various problems of the dynamics of mechanical systems and included this division in his original plan of exposition of all mechanics. To explain this inconsistency, we will recall that a general method for investigating mechanical systems was proposed by d'Alembert in his 1743 publication **Traité de dynamique**. Without going into details, we will point out that, as early as in the 1730s, Euler had at his disposal the prerequisites for a fairly general method, equivalent to d'Alembert's principle, and subsequently developed his own approach to problems of the dynamics of mechanical systems in the spirit of Newton's ideas, quite independently. It appears that it was for this reason that Euler never referred to "d'Alembert's principle" and at the same time never set out his own general approach as a separate method, in order not to enter into an argument with d'Alembert about priority.

We note one particular result of Euler that relates to the mechanics of relative motion, set out in his essay **De motu corporum in superficiebus mobilibus** (E.86; 1746), where Euler obtained for the first time, simultaneously with D. Bernoulli, the law of conservation of angular momentum. In solving problems, Euler systematically calculated the absolute accelerations in combined motion (in a horizontal plane), breaking them down along the directions of relative and translational motion. In these breakdowns, the relative, translational and additional (Coriolis) accelerations are easily discerned. Similar breakdowns of absolute accelerations were widely used by Euler in later investigations of the flow of water in rotating pipes. However, he did not detect in the structure of these breakdowns a common property, subsequently found by Coriolis. An example of the differential equations of motion of a system of n bodies appears in Euler's work in the second half of the 1740s in an investigation of the problem of the motion of a system of hinged rigid rods on a smooth horizontal plane (published in 1751).

The development of the mechanics of systems with a finite number of degrees of freedom was closely connected in the first half of the eighteenth century with the oscillation theory. Without dwelling on this division of dynamics, important for the history as a whole, it must be noted that in the 1730s, and at the beginning of the 1740s, Euler (along with D. and J. Bernoulli) made a considerable contribution to its development. Of great importance to the development of the oscillation theory was his discovery of a method for integrating linear differential equations with constant coefficients. In particular, as early as 1739, for the case of a sinusoidally excited oscillator, Euler discovered the phenomenon of resonance (published in 1750).

From the middle of the 1740s, Euler carried out many investigations in celestial mechanics. Of most importance here were various aspects of three-body problems – the theory of the motion of the moon, the theory of perturbation of planetary motions and, finally, from the 1760s onwards, the three-body problem itself in pure form. Besides this, he, of course, also investigated methods for determining unperturbed orbits, including the orbits of comets for which he obtained, for example, the well-known equation expressing a time interval in terms of the sum of radius vectors and the chord, which remained the basis, until recently, for calculating parabolic orbits. Euler's contribution to the founding and development of celestial mechanics is very significant. However, in the modern literature, Euler's methods are often linked with the names of other scientists, who only refined his methods. Furthermore, it must be said that the history of celestial mechanics in the eighteenth century, generally speaking, has hitherto not been studied sufficiently. Normally, as early as in the nineteenth century, researchers did not refer to sources that came before Laplace's **Traité de mécanique céleste**, which generalized the main advances of that time in this area. Nevertheless, Laplace himself highly valued Euler's contribution to celestial mechanics, especially his contribution to the theory of the motion of planets.

In fact, it can be considered that the theory of perturbations of planetary motions dates back to Euler's memoir concerning inequalities in the motion of Jupiter and Saturn (presented in a competition of the Parisian Academy of Sciences in 1747 and published in 1749). This memoir, together with d'Alembert's treatise concerning the precession of the equinoxes and the nutation of the Earth's axis (published in Paris in the same year) and the **Téorie de la Lune** by Clairaut (presented in a competition of the St Petersburg Academy of Sciences in 1750 and published in 1752), can justifiably be considered to be the starting point of all modern celestial mechanics (the sources of which date back, of course, to Newton's **Principia**). Note that this and other studies by Euler at the end of the 1740s already contain the idea of the method of variation of elements, which he developed in more detail in subsequent work.

It is curious that in the 1740s Euler began to doubt the strict validity of Newton's law of gravitation. He was brought to doubt this both by certain general deliberations and by errors that he allowed in calculations of planetary perturbations. And it was only the investigation by Clairaut (in the essay just mentioned) of the motion of the moon's apogy that convinced Euler of the accuracy of Newton's law. Subsequently, Euler continued his investigations of the motion of Jupiter and Saturn, having presented his next memoir on this theme to the Paris Academy in 1752. As underlined by M. F. Subbotin: "the idea he developed here of finding perturbed values of eccentricities and longitudes of perihelions . . . is essentially the embryo of the theory of the representation of secular perturbances in trigonometric form", developed later by Lagrange. Subsequently, Euler introduced many other improvements into perturbation theory. For example, he developed an effective numerical method for integrating equations of perturbed motions in rectangular coordinates. We also note another large range of investigations by Euler on the transfer to the theory of planetary perturbations of powerful methods developed in the theory of the motion of the moon.

The first investigations Euler made on the motion of the moon were summed up in his Berlin monograph (E.187; 1753). On the basis of the theory developed here, Mayer's tables of the moon were calculated, which were awarded a monetary prize by the British parliament (with payment of one-tenth of the total to Euler). However, Euler's second theory of the motion of the moon (E.418; 1772), published in Petersburg, is the most interesting. Originally it did not attract the wide attention of astronomers owing to its complexity. However, 100 years later, interest in this theory was shown by Hill, who developed the ideas set out in the theory and published two papers in 1877 and 1878 that became, in the words of M. F. Subbotin, "one of the most important sources for further progress of all celestial mechanics". The equations of motion of the moon that were written by Euler were typical of the theory of non-linear vibrations, and his investigations of methods of integrating them that were continued by Hill made a significant contribution to the general theory of non-linear vibrations.

Euler's participation in the initial development of the first integral variational principle of mechanics – the principle of least action, originally suggested in an unclear but pretentious form by the president of the Berlin Academy, Maupertuis – must be noted. Euler was responsible for essentially the first rigorous formulation of this principle for the motion of a point mass, together with the body of mathematics for calculus of variations he developed. However, the principle of least action in the form introduced by Euler was still unsuitable for solving new problems in mechanics. Subsequent progress was achieved after this principle was extended by Lagrange to mechanical systems. After this, in the nineteenth century, came the development of classical integral variational principles that, ultimately, exceeded the limits of mechanics itself.

4. The motion of a rigid body

An outstanding contribution was made by Euler to the creation of the general theory of the motion of a rigid body. Originally, at the end of the 1730s, in the preparation of his **Scientia navalis** (E.110,111; 1749), printed in Petersburg, he examined certain particular problems of the dynamics of rigid bodies. In this large two-volume work, we find the breakdown of the motion of a ship into translational and rotational motion, an attempt to calculate the small perturbations of a ship on water, the progressive science of the stability of equilibrium of floating bodies, and elements of the science of moments of inertia.

Euler returned to the general theory of the motion of a rigid body in 1749–1750. A known incentive behind this seems to have been investigations by d'Alembert that went into his treatise on the precession of the equinoxes and the nutation of the Earth's axis and contained certain approaches to the theory of the rotation of a rigid body. The first key step in constructing the dynamics of a rigid body was performed by Euler in his memoir **Decouverte d'un nouveau principe de mécanique** (E.177; 1752), where a "general and fundamental principle of all mechanics" was set out. This essentially consisted of the application of the principal law of dynamics (Newton's second law) for each infinitesimal particle of a body in projections onto an axis of a stationary system of coordinates.

$$Md^2x = Pdt^2, \quad Md^2y = Qdt^2, \quad Md^2z = Rdt^2$$

where M is the mass of the particle, and P, Q and R are the components of external forces (Euler writes these equations with a coefficient of 2 on the left-hand side, which is due to the system of physical units that was used then, in which the accelerations were dimensionless and the velocities were measured in a special way). In his memoir, he wrote that "it is upon this single principle that all other principles must be based, both those already obtained in mechanics and hydraulics and those that are now used to determine the motions of rigid and fluid bodies, and those that are still unknown and that we need in order to develop both the cases of rigid bodies that have been indicated above and many others that relate to fluid bodies".

Thus, Euler's new principle included the isolation of an elementary particle from a continuum and the application to it of Newton's fundamental law written in projections onto the axes of a stationary system of coordinates. It is now difficult to imagine the impetus that was given to mechanics by this work, which today seems to us to be self-evident. But it was this work that opened up the simplest and most natural way of constructing the mechanics of a rigid body and, most importantly, continuum mechanics. For the sake of accuracy it must be pointed out that the representation of the fundamental law of dynamics in projections onto the axes of a stationary system of coordinates in order to study the motion of a point mass was proposed as an independent "principle" of mechanics by Maclaurin as far back as 1742 in his "Treatise of Fluxions". In the 1740s, such representation of the equations of motion had already been used by a number of scientists, in particular J. Bernoulli, Clairaut, d'Alembert and Euler himself. However, before Euler it had not entered anyone's head that these differential equations, being written for an arbitrary element of a medium or body, lead directly to a general mathematical formulation of mechanics problems. (The need for the independent use of the law of angular momentum was, it seems, realized by Euler much later.) On the basis of this approach, Euler immediately derived general equations of the rotation of a rigid body, but he presented them initially in an inconvenient form for investigation, related to a stationary system of coordinates, introducing the moments of inertia of the body (with respect to stationary axes) that change during the motion of the body.

In 1755, Segner published a short essay devoted to the investigation of the free axes of rotation of arbitrary bodies. The concept of the free axis of rotation had been used by Euler earlier in his **Scientia navalis**, but there he still did not accept that each body, as established by Segner, has three mutually perpendicular axes of free rotation. As Euler acknowledged, it was becoming acquainted with Segner's work that prompted him to return to the study of the rotation of rigid bodies and provided him with a pointer for constructing a compact general theory. As a result of his studies on the theory of the rotation of rigid bodies, dating back to the end of the 1750s (but published in the **Mémoires** of the Berlin Academy only in 1765), Euler employed, as the basic system of coordinates, the principal axes of inertia of the body, which are the free axes of rotation, and gave to the general dynamic equations the classical form that remains to this day (apart from the notation):

$$dx + \frac{c-b}{a}yzdt = \frac{Pdt}{Ma}, \quad dy + \frac{a-c}{b}xzdt = \frac{Qdt}{Mb}, \quad dz + \frac{b-a}{c}xydt = \frac{Rdt}{Mc}$$

where M is the mass, a, b and c are the principal central moments of inertia of the body (denoted by Euler as aa, bb and cc), and P, Q and R are the moments of external forces (he writes these equations with a coefficient $2g$ on the right-hand side, which is due to the use, already mentioned above, of a system of physical units different from the modern system). Euler then investigated the first celebrated case of integrability in the problem of the rotation of a rigid body about a stationary point – the centre of mass. Finally, he is responsible for the development of the kinematics of a rigid body, including the derivation of both forms of kinematic equation of rotation (one of which is sometimes called Poisson's equation), and equally the expanded science of moments of inertia (the geometry of masses), with the exception, however, of the construction of the ellipsoid of inertia.

The completion of the main stage of his investigations into the dynamics of a rigid body was his treatise **Theoria motus corporum solidorum seu rigidorum** (E.289; 1765), which he finished in 1760 and counted as the third volume of his **Mechanica**. Euler continued to study the dynamics of a rigid body in subsequent years. In particular, in his essay **Nova methodus corporum rigidorum determinandi** (E.479; 1776), for the first time the six equations of motion of an arbitrary body that comprise the law of momentum and the law of angular momentum were written out

$$\int dM\frac{d^2x}{dt^2} = P, \quad \int zdM\frac{d^2y}{dt^2} - \int ydM\frac{d^2z}{dt^2} = S$$

$$\int dM\frac{d^2y}{dt^2} = Q, \quad \int xdM\frac{d^2z}{dt^2} - \int zdM\frac{d^2x}{dt^2} = T$$

$$\int dM\frac{d^2z}{dt^2} = R, \quad \int ydM\frac{d^2x}{dt^2} - \int xdM\frac{d^2y}{dt^2} = U$$

Euler wrote these equations with an additional coefficient on the right-hand side, as in the system of equations given above. The well-known historian of mechanics C. Truesdell considers this essay by Euler to be the first appearance in the history of mechanics of both these laws as "fundamental, general and independent laws of mechanics for all kinds of motion of all kinds of bodies". In this connection, Truesdell suggested that the combination of these laws of mechanics should be called Euler's laws.

5. Fluid mechanics

The development of the fundamental principles of fluid mechanics is largely due to Euler. His interest in problems of flow emerged in his youth. Under the influence of J. Bernoulli, when investigating the discharge of liquid from vessels, he employed the principles of the conservation of energy, using, along with this, the hypothesis, used earlier, of plane sections and a form of the law of continuity corresponding to it. Euler presented his results to the Petersburg Academy at the age of 20 in August 1727, 2 weeks after a similar paper by D. Bernoulli. The results of the two authors tallied, and in this delicate situation Euler conceded the right to publication of the results obtained to his senior colleague, ceasing his own studies in this area for a quarter of a century. He returned to general problems of flow only at the beginning of the 1750s, after the publication of **Hydrodynamica** by D. Bernoulli (1738) and **Hydraulica** by J. Bernoulli (1743). By this time, Euler had finally developed two concepts necessary for the general construction of hydrodynamics: a clear concept of the pressure in a flowing fluid and a simple formulation of the principal law of dynamics (the law of momentum) for an elementary particle of a medium. Euler's definition of pressure was the final refinement of the evolution of this concept, which was developed in 1730 by D. Bernoulli and partly improved upon in **Hydraulica** by J. Bernoulli.

The first attempts to derive general continuous equations of flow were undertaken at the very end of the 1740s by d'Alembert. He presented his hydrodynamic investigations at the end of 1749 in Berlin and later published them in Paris in 1752. Besides considerations of the resistance of fluids, his essay contained an examination of the continuous velocity field and a derivation of partial differential equations describing flow in certain cases. Although full of new ideas, d'Alembert's essay did not extend to general equations of flow. Moreover, written in the unclear and inconsistent style peculiar to d'Alembert, this essay is difficult to read and understand, which was pointed out, in particular, by Truesdell. However, these shortcomings of d'Alembert's work did not, of course, nullify its value, especially for Euler, who had the opportunity to become acquainted with it in 1750.

It was Euler who managed, with characteristic clarity and accuracy, to construct the entire system of equations of the continuous flow of an ideal fluid. He based this on his "new principle of mechanics", mentioned above. His first results on the general theory of the fluid flow date back, it appears, to 1752. His two main fundamental essays on hydrostatics and hydrodynamics, dating back to 1753–1755, were published in 1757 in the eleventh volume of his **Mémoires** by the Berlin Academy.

In the first of these essays (E.255; 1757), Euler extended Clairaut's results and gave the exposition of hydro- and aerostatics a form that has largely been retained to this day. He introduced the concept of the pressure p measured by the height of a column of homogeneous fluid, pointed out the dependence of pressure at least on the density and temperature and then derived a general equation for the fluid equilibrium.

$$dp = q(Pdx + Qdy + Rdz)$$

where p is the pressure, which Euler understood as the height of the column of homogeneous fluid, and q is the dimensionless density; the components of the mass forces relate to the acceleration due to gravity. The system of units used here differs slightly from that employed by Euler in his original exposition of the "new principle of mechanics". Euler then introduced the concept of the potential of forces s and, having rewritten the general equation of equilibrium in the form $dp = qds$, pointed out the constancy of pressure, density and temperature on the surfaces of the level of potential s. He then derived general relations for the case of an ideal gas, examined the forces acting on an immersed body and proceeded to a detailed examination of various cases of the equilibrium of liquids and gases. Here, he obtained, in particular, the well-known barometric formula for an isothermal atmosphere, and also proposed a physical notion of temperature: at a constant volume, it is advisable to consider the temperature to be proportional to the pressure of the gas.

Euler's second essay **Principes généraux du movement des fluides** (E.226; 1757) began with a general formulation of problems of the theory of the flow of an ideal fluid. Then, from a normal (for our time) examination of an elementary fluid parallelepiped, general equations of hydrodynamics and an equation of continuity for compressible fluids were derived. Euler also states here that to these four equations a fifth must be added that gives the relation between the pressure, density and an additional physical quantity that affects the pressure, and that, generally speaking, implies the temperature. The five equations thus obtained, says Euler, "contain the entire theory of the fluid flow".

After deriving the principal equations of hydrodynamics, Euler introduces the potentials of forces S and velocity W and obtains the formula

$$dp = q(dS - d\Pi - udu - \upsilon d\upsilon - wdw), \quad \Pi = \frac{\partial W}{\partial t}$$

and the corresponding integrals for the case of an incompressible fluid, and also, broadly for barotropic processes, the integrals that in today's parlance are normally known as the Lagrange–Cauchy integrals. Euler stipulates specially here the existence of non-potential flows, citing as an example one case of the vortical rotation of an incompressible fluid when there are no mass forces. The essay ends with an investigation of individual special cases of flow and with the remark that the equations derived transfer problems of flow from the area of mechanics to the area of mathematical analysis. In reading this essay, the clarity and simplicity of the expression of Euler's thoughts (characteristic of most of his other work) are particularly striking. It is difficult, occasionally, to believe that he is separated from us by two and a half centuries.

Euler's first three works on fluid mechanics were followed by many other essays devoted to hydrodynamics and the theory of sound propagation. Their completion and general conclusion was a large work (516 pp.) dating back to the end of the 1760s and published in four parts in 1769–1772 in **Novi Commentarii** by the Petersburg Academy of Sciences. Its first part includes an examination of the general properties of liquids and gases, the derivation of the general equations of equilibrium and an investigation of special cases of equilibrium in a gravity field and a field of central forces. In the second part, a system of general equations is derived for the dynamics of an ideal fluid, and cases of the flow of an incompressible fluid, including potential flow, are examined in more detail. The final chapter is devoted to determining the fluid flow by a prescribed initial state; here, in particular, general equations of hydrodynamics are derived in so-called Lagrangian variables – material variables. Note that these variables were pointed out to Lagrange by Euler in his letter of 1 January 1760, published by Lagrange in 1762 together with his own associated investigations. In the third part of the work, Euler examines flow in tubes of constant and variable cross-section, and calculates the rise in water level using pumps and flow under the action of a temperature difference. The last part is a general conclusion of his many preceding investigations into acoustics and the theory of musical wind instruments.

Thus, Euler laid the foundations of the entire dynamics of an ideal fluid, with the exception of ultrasonic aerodynamics, which was conceived a century later and developed in the twentieth century. Without possessing a general concept of stress, introduced by Cauchy in 1823, Euler was unable, of course, to move on to study more complex models of continuum mechanics – viscous fluids and elastic bodies. However, he did much to lay the foundations for the subsequent development of continuum mechanics.

6. The mechanics of flexible and elastic bodies

We will now dwell on Euler's investigations of the mechanics of flexible and elastic bodies. Euler was interested in problems of the mechanics of elastic bodies (rods) from his early days. It is curious that, in one short note written by Euler while still in Basel, but only published posthumously in 1862, Truesdell discovered the first derivation of Jacob Bernoulli's law of the bending of rods from Hooke's law for the stretching of fibres – a result not mentioned by Euler himself and then reopened by him again much later. Without dwelling on important studies by Euler (and D. Bernoulli) on the transverse vibrations of rods, we will pass on to celebrated investigations he made into equilibrium forms of elastic rods and their buckling. These investigations were initiated by the discovery by D. Bernoulli (in 1742) of the extremality property of the elastic energy of bent elastic rods. Classical results of Euler that date back to then were published by him in 1744 in the form of the special appendix **De curvis elasticis** to his treatise on the variational calculus (E.65; 1744). Here, he analysed nine possible types of equilibrium form of an originally rectilinear rod of rectangular cross-section, bent under the action of a force and moment applied to its ends. This also contains essentially a general equation for the critical force during the buckling of a rod. Euler himself, by the way, used this formula for the case of a rod with hinged ends. Subsequently, he returned repeatedly to the question of the buckling of columns, and his final investigations in this area, dating back to the end of the 1770s, deal with the buckling of columns under their own weight. In the first part of the **Acta** of the Petersburg Academy in 1778 there are three works by Euler on this problem, in which he successively overcame the difficulties encountered, ultimately arriving at the correct solution.

Euler took an active part in the discussion of the vibrations of a string. Essentially, the problem of small transverse vibrations of a string (and of the propagation of sound) was the first problem of dynamics of systems with an infinite number of degrees of freedom. It is remarkable that this problem began to be studied so long before the dynamics of systems with a finite number of degrees of freedom was developed. The classical wave equation of the vibrations of a string was obtained in 1746 by d'Alembert (published in 1749); then he also found his solution containing two arbitrary functions of the arguments $(ct + x)$ and $(ct - x)$. However, d'Alembert arbitrarily limited the class of functions occurring in the solution of the wave equation by certain conditions of "continuity" and "smoothness". Euler studied the wave equation immediately after d'Alembert and underlined that the general solution of the problem of a string should include functions of a much wider class – arbitrary piecewise-smooth functions. The third active participant in the discussion of the vibrations of a string – D. Bernoulli – was likewise engaged in it almost from the very start. Bernoulli took exception to the abstract discourses of d'Alembert and Euler on arbitrary functions and believed that the vibrations of a string is presented more simply and naturally as the superposition of simple harmonic vibrations. The discussion concerning the nature of the solutions of the wave equation continued for many years (Lagrange joined in later) and had a considerable influence on the subsequent development of the methods of mathematical physics, and, to a certain extent, the theory of functions of a real variable.

We will again mention Euler's generalizing investigations dating back to the 1770s on the mechanics of flexible and elastic (one-dimensional) bodies. Here, he obtained general equations of the equilibrium and motion of a deformable line (and plane) without special assumptions concerning the nature of its material and the smallness of the strains. He examined the transverse forces acting in cross-sections, anticipating the concept of shear stresses. Finally, his introduction of a material physical characteristic fully equivalent to Young's modulus, and thereby the separation in problems of elasticity theory of the elastic properties of a material from the shape of the body under examination, dates back to these years.

7. Conclusions

In this outline, Euler's essays on applied mechanics have hardly been mentioned. Historically, of course, they are second to his investigations in the area of mathematics and rational mechanics, but in the development of applied mechanics, too, he left a profound mark. He examined problems of dry friction (in particular, a formula for calculating the friction of a rope wrapped about a round shaft is named after him). He carried out interesting work on the general theory of machinery, and also on designing different specific machines, mechanisms and instruments (for example, balances). His investigation of the shape of gear-wheels deserves special mention. He devoted a cycle of studies to hydraulic motors, and, in particular, to Segner's wheel theory – the prototype of the hydraulic jet engine.

He carried out extensive investigations on ship theory. After publishing the aforementioned two-volume **Scientia navalis** (E.110,111; 1749), he studied various propulsion systems, including water-jet propulsors, deriving for the latter certain mathematical formulae that retain their value to this day. He also obtained certain results on ship structural mechanics. Finally, in 1773, he published in Petersburg a practical guide, written in French, on shipbuilding and navigation. It is remarkable that this guide was then republished in Paris and used there as a textbook, and was also translated into English, Italian, and Russian.

Returning to the programme of the construction of mechanics proposed by Euler in his youth, it must be pointed out that, over the entire course of his life, he constructed three out of six planned general divisions of mechanics: analytically set out point mechanics, rigid body mechanics and hydrodynamics. He made a fundamental contribution, along with other scientists, to studies of flexible bodies and to systems mechanics. As regards the theory of elasticity, to which he devoted a number of important investigations, it was created only in the nineteenth century. It can be said that Euler managed brilliantly the grandiose programme that he set himself in the first volume of **Mechanica** (1736), not then realizing its improbable difficulty. More than to any other, it is to Euler that we are indebted for clarifying the principles of mechanics.

The total volume of essays he wrote is huge. His published scientific studies, numbering over 800, comprise about 30 000 printed pages and consist of 600 papers in periodicals and the collections of the Petersburg Academy of Sciences, 130 papers in various European journals, 15 memoirs, which were awarded prizes and promoted by the Paris Academy of Sciences, and 40 books of individual essays. A century ago – on the initiative of Swiss mathematicians and with the support of three leading European academies of science (Petersburg, Berlin, and Paris) – the publication of Euler's "Complete Works" (*Opera omnia*) was undertaken. The first volume was published in 1911, and the publication of the final two out of a planned 72 volumes is expected in the next 2–3 years. His essays are printed here in the original language, i.e. mainly in Latin or French. A decision to publish an additional series of his **Opera omnia** that would contain his scientific correspondence was taken by an agreement between the Swiss Academy of Natural Sciences and the Russian Academy of Sciences in the 1970s. Of the planned ten volumes of this series, only four have so far been published.

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